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Convergent Sequences of Orthogonal Polynomials

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I. INTRODUCTION

Let $\{P_n(x)\}$ be a sequence of real orthogonal polynomials whose zeros have a finite lower bound. If the associated Hamburger moment problem is indeterminate, then there exist constants k_n such that $\{k_nP_n(z)\}$ converges uniformly on bounded sets to an entire function with real, simple zeros. The converse is not in general true and it seems natural to ask what implications there are in the convergence of $\{k_nP_n(z)\}$.

In this paper we will consider this question of the convergence of a sequence of orthogonal polynomials restricting our work to the case where the true interval of orthogonality is a half line. Our main interest will be in relating the convergence of the sequence of orthogonal polynomials to the behavior of the coefficients in the three term recurrence relation satisfied by these polynomials. Except for the "symmetric case," $P_n(-x) = (-1)^n P_n(x)$, when conclusions can be obtained as more or less routine corollaries to results concerning the case where the interval of orthogonality is a subset of $(0, \infty)$, the situation where the true interval of orthogonality is $(-\infty, \infty)$ is left completely open.

Specifically, then, we consider a sequence $\{P_n(x)\}$ of monic polynomials defined by a recurrence:

$$\begin{gathered} P_n(x) = (x - c_n) P_{n-1}(x) - \lambda_n P_{n-2}(x) \\ P_{-1}(x) = 0, \qquad P_0(x) = 1, \qquad c_n \text{ real}, \qquad \lambda_{n+1} > 0 \qquad (n \geqslant 1). \end{gathered} \tag{1.1}$$

By Favard's Theorem [9], there is a distribution function ψ (= bounded, non-decreasing function) with an infinite spectrum, $\mathfrak{S}(\psi)$ (= support of $d\psi(x)$), such that

$$\int_{-\infty}^{\infty} P_m(x)\, P_n(x)\, d\psi(x) = \lambda_1\lambda_2 \cdots \lambda_{n+1}\delta_{mn}, \tag{1.2}$$

where

$$\lambda_1 = \mu_0 = \int_{-\infty}^{\infty} d\psi(x).$$

If the associated Hamburger moment problem is determined, then (1.2) uniquely determines ψ up to an additive constant at all points of continuity.

Now let $x_{n1} < x_{n2} < \cdots < x_{nn}$ denote the zeros of $P_n(x)$. By the well-known separation property of the zeros of orthogonal polynomials, $\{x_{ni}\}_{n=i}^{\infty}$ is a decreasing sequence and we put

$$\xi_i = \lim_{n\to\infty} x_{ni} \qquad (i = 1, 2, 3,\ldots),$$

and we note that

$$-\infty \leqslant \xi_1 \leqslant \xi_2 \leqslant \xi_3 \leqslant \cdots.$$

We consider throughout this paper the case $\xi_1 > -\infty$. That is, we assume equivalently that there exists a real c such that $c_n > c$ $(n \geqslant 1)$ and $\{\lambda_{n+1}/[(c_n - c)(c_{n+1} - c)]\}_{n=1}^{\infty}$ is a chain sequence (and ξ_1 is the largest value of c for which this holds) (See [5]; for chain sequences, see Wall [15]).

II. Preliminary Results

We write

$$\begin{aligned} X &= \{x_{ni} \mid i = 1,\ldots, n;\, n = 1, 2, 3,\ldots\},\\ \Xi &= \{\xi_i \mid i = 1, 2, 3,\ldots\}, \\ Z &= X' \cup \{x \mid P_n(x) = 0 \text{ for infinitely many } n\}, \end{aligned} \tag{2.1}$$

(where X' denotes the derived set).

Then we clearly have that

$$\Xi \subset Z$$

and that ξ_1 is the least member of both Ξ and Z. Also, according to a theorem of Stone [13, Theorem 10.42],

$$\mathfrak{S}(\psi) \subset Z,$$

whenever the Hamburger moment problem is determined.

Lemma 1. $(\xi_i, \xi_{i+1}) \cap Z = \varnothing$ $(i \geqslant 1)$.

Proof. If (ξ_i, ξ_{i+1}) is not an empty interval, let $\xi_i < \alpha < \beta < \xi_{i+1}$. Since $x_{nk} \downarrow \xi_k$ $(n \to \infty)$, it follows that $x_{nj} \in (\alpha, \beta)$ for at most finitely many n and j. Thus $(\alpha, \beta) \cap Z = \varnothing$.

THEOREM 1. (i) *If* $\lim_{n\to\infty} \xi_n = +\infty$, *then* $Z = \Xi$.

(ii) *If* $\xi_p = \xi_{p+1}$ *for some* p, *then the associated Hamburger moment problem is determined,* ξ_p *is a limit point of* $\mathfrak{S}(\psi)$, *and* $\xi_{p+k} = \xi_p$ $(k = 1, 2, 3,...)$.

Proof. (i) By Lemma 1 and the fact that ξ_1 is the least member of Z,

$$\Xi^c \cap Z = \bigcup_{i=0}^{\infty} (\xi_i, \xi_{i+1}) \cap Z = \varnothing \qquad (\xi_0 = -\infty).$$

Thus $Z \subset \Xi$. Since always $\Xi \subset Z$, (i) follows.

(ii) For $n > p$, $\mathfrak{S}(\psi) \cap (x_{np}, x_{n,p+1}) \neq \varnothing$ [14, Theorem 3.41.2]. Therefore, if $\xi_{p+1} = \xi_p$, ξ_p must be a limit point of $\mathfrak{S}(\psi)$ (where ψ is *any* solution of the moment problem). Now if this moment problem is indeterminate, then according to Stone [13, Theorem 10.42] there exist solutions of the moment problem whose spectra do not have ξ_p as a limit point. Thus the moment problem must be determined, hence $\mathfrak{S}(\psi) \subset Z$. In particular, ξ_p is a limit point of Z.

Now suppose there is a least integer $q > 1$ such that $\xi_p < \xi_{p+q}$. According to Lemma 1, $(\xi_p, \xi_{p+q}) \cap Z = \varnothing$ while $(\xi_1, \xi_p) \cap Z$ is finite. That is, ξ_p would not be a limit point of Z. This contradiction shows that $\xi_{p+k} = \xi_p$ for all $k > 0$.

We now turn our attention to the polynomials themselves.

LEMMA 2. *If* $0 < x \leqslant \xi_1$, *then* $\{P_n(x)/P_n(0)\}_{n=1}^{\infty}$ *is a decreasing sequence and*

$$\frac{P_n(x)}{P_n(0)} > \prod_{i=1}^{n} \left(1 - \frac{x}{\xi_i}\right) \geqslant 0. \tag{2.2}$$

Proof. Since $0 < x < x_{n+1,i} < x_{n,i}$,

$$\frac{P_n(x)}{P_n(0)} = \prod_{i=1}^{n} \left(1 - \frac{x}{x_{ni}}\right) > \prod_{i=1}^{n} \left(1 - \frac{x}{x_{n+1,i}}\right) > \frac{P_{n+1}(x)}{P_{n+1}(0)}.$$

Thus $\{P_n(x)/P_n(0)\}$ is decreasing and (2.2) follows from $0 < \xi_i < x_{ni}$.

LEMMA 3. *If* $0 < x < \xi_1$, *then*

$$E(x) \equiv \lim_{n\to\infty} \frac{P_n(x)}{P_n(0)} > 0$$

if and only if

$$\sum_{i=1}^{\infty} \xi_i^{-1} < \infty.$$

Proof. Because of (2.2), it is clear that $\sum \xi_i^{-1} < \infty$ implies

$$E(x) \geqslant \prod_{i=1}^{\infty} \left(1 - \frac{x}{\xi_i}\right) > 0, \qquad 0 < x < \xi_1 .$$

Conversely, for arbitrary $\epsilon > 0$ and positive integer i, there exists $N_i = N_i(\epsilon)$ such that

$$x_{ni} < \xi_i + \epsilon \qquad \text{for } n \geqslant N_i .$$

Hence for any integer $K > 0$, $0 < x < \xi_1$,

$$\prod_{i=1}^{K} \left(1 - \frac{x}{x_{ni}}\right) < \prod_{i=1}^{K} \left(1 - \frac{x}{\xi_i + \epsilon}\right), \qquad n > N = \max(N_1, ..., N_k).$$

Thus

$$\frac{P_n(x)}{P_n(0)} < \left[\left(1 - \frac{x}{x_{n,K+1}}\right) \cdots \left(1 - \frac{x}{x_{nn}}\right)\right]^{-1} \frac{P_n(x)}{P_n(0)} < \prod_{i=1}^{K} \left(1 - \frac{x}{\xi_i + \epsilon}\right),$$

whence

$$0 \leqslant E(x) < \prod_{i=1}^{K} \left(1 - \frac{x}{\xi_i + \epsilon}\right).$$

Thus if $E(x) > 0$, then $\prod_i (1 - x/\xi_i)$ converges; hence $\sum_i \xi_i^{-1} < \infty$.

Henceforth, we will use the convention that $\sum_n' a_n^{-1}$ denotes the series obtained after omitting from $\{a_n\}$ any terms that vanish.

THEOREM 2. *Let $\xi_1 > -\infty$. Then $\sum_i' \xi_i^{-1}$ converges if and only if for any $a < \xi_1$, $\{P_n(z)/P_n(a)\}_{n=0}^{\infty}$ converges uniformly on bounded sets to an entire function whose zeros are simple and are precisely the points ξ_i $(i \geqslant 1)$.*

Proof. Since we can consider, if necessary, $Q_n(z) = P_n(z + a)$, there is no loss of generality if we assume $a = 0 < \xi_1$.

Suppose therefore that $\xi_1 > 0$ and $\sum \xi_i^{-1} < \infty$. Then by Theorem 1, $\xi_i < \xi_{i+1}$. For $|z| \leqslant R$,

$$\left|\frac{P_n(z)}{P_n(0)}\right| \leqslant \prod_{i=1}^{n} \left(1 + \frac{|z|}{x_{ni}}\right) \leqslant \prod_{i=1}^{\infty} \left(1 + \frac{R}{\xi_i}\right);$$

so $\{P_n(z)/P_n(0)\}$ is uniformly bounded on $|z| \leqslant R$. In view of Lemma 3, the Stieltjes–Vitali theorem shows that $\{P_n(z)/P_n(0)\}$ converges uniformly on $|z| \leqslant R$ to an entire function $E(z)$ that is not identically zero.

Since $x_{ni} \to \xi_i$, it follows from the uniform convergence that $E(\xi_i) = 0$. On the other hand, $P_n(x)$ has only simple zeros and $\xi_i < \xi_{i+1}$. Hence by Hurwicz' Theorem, $E(z)$ has only simple zeros and no zeros other than the ξ_i.

The converse is contained in Lemma 3.

III. The Recurrence Relation and Convergence

We next investigate the connection between the convergence of $\{P_n(z)/P_n(a)\}$ and the coefficients in (1.1), still maintaining our hypothesis, $\xi_1 > -\infty$.

Let us now put

$$\alpha_n(x) = \frac{\lambda_{n+1}}{(c_n - x)(c_{n+1} - x)} \qquad (n \geqslant 1), \tag{3.1}$$

and note the identity

$$\alpha_n(x) = [1 - m_{n-1}(x)]\, m_n(x) \qquad (n \geqslant 1), \tag{3.2}$$

where

$$m_n(x) = 1 + \frac{P_{n+1}(x)}{(c_{n+1} - x)\, P_n(x)} \qquad (n \geqslant 0). \tag{3.3}$$

We then have that $x \leqslant \xi_1$ if and only if $x < c_n$ $(n \geqslant 1)$ and $\{a_n(x)\}_{n=1}^{\infty}$ is a chain sequence whose minimal parameter sequence is $\{m_n(x)\}_{n=0}^{\infty}$ (see [7, p. 364]).

From (3.3), we have the identities

$$\frac{P_n(x)}{P_n(a)} = \frac{P_n(x)}{P_{n-1}(x)} \frac{P_{n-1}(a)}{P_n(a)} \frac{P_{n-1}(x)}{P_{n-1}(a)}$$

$$= \frac{[1 - m_{n-1}(x)]\,(x - c_n)}{[1 - m_{n-1}(a)]\,(a - c_n)} \cdot \frac{P_{n-1}(x)}{P_{n-1}(a)} \qquad (n \geqslant 1),$$

$$\frac{P_n(x)}{P_n(a)} = \prod_{k=1}^{n-1} \frac{1 - m_k(x)}{1 - m_k(a)} \prod_{i=1}^{n} \frac{c_i - x}{c_i - a}. \tag{3.4}$$

Theorem 3. *Let $\xi_1 > -\infty$. Then $\sum_i' \xi_i^{-1}$ converges if and only if $\sum_n' c_n^{-1}$ converges and $\prod_{k=1}^{\infty} m_k(x)/m_k(a)$ converges for some (hence all) $a < x < \xi_1$.*

Proof. For any $a < x \leqslant \xi_1$, $\alpha_n(a) < \alpha_n(x)$, hence $0 < m_n(a) < m_n(x) < 1$ $(n \geqslant 1)$ [15, Theorem 19.6]. Further, $0 < c_n - x < c_n - a$ since $c_n > \xi_1$. It follows that both products on the right side of (3.4) are positive and decreasing functions of n. Thus $\{P_n(x)/P_n(a)\}$ converges to a positive limit if and only if the infinite products

$$\text{(i)} \quad \prod_{k=1}^{\infty} \frac{1 - m_k(x)}{1 - m_k(a)} \qquad \text{and} \qquad \text{(ii)} \quad \prod_{n=1}^{\infty} \frac{c_n - x}{c_n - a}$$

both converge.

Writing

$$\frac{m_k(x)}{m_k(a)} = \frac{1 - m_{k-1}(a)}{1 - m_{k-1}(x)} \frac{\alpha_k(x)}{\alpha_k(a)} = \frac{1 - m_{k-1}(a)}{1 - m_{k-1}(x)} \frac{(c_k - a)(c_{k+1} - a)}{(c_k - x)(c_{k+1} - x)}$$

we see that (i) and (ii) both converge if and only if $\prod m_k(x)/m_k(a)$ and (ii) converge. Finally, we note that (ii) converges if and only if $\sum' c_n^{-1}$ converges. Reference to Theorem 2 now completes the proof.

COROLLARY. *If* $\xi_1 > -\infty$ *and* $\sum_n' c_n^{-1} = \infty$, *the associated Hamburger moment problem is determined.*

Proof. If the Hamburger moment problem is indeterminate there exist constants k_n such that $\{k_n P_n(z)\}$ converges uniformly on bounded sets to an entire function whose zeros are real and simple (see for example [8, p. 479]).

The criterion in the above corollary is less general than the well-known condition for determinacy due to Carleman, viz., $\sum \lambda_n^{-1/2} = +\infty$, in the sense that Carleman's criterion does not require that ξ_1 be finite. On the other hand, it is not contained in Carleman's criterion as the following example shows.

Let

$$c_{2n} = n, \qquad c_{2n+1} = n^2, \qquad \lambda_{n+1} = \tfrac{1}{4} c_n c_{n+1} \qquad (n \geqslant 1).$$

Then Carleman's criterion yields no conclusion about the determinacy of the moment problem. However, $\alpha_n(0) = \frac{1}{4}$; so $\{\alpha_n(0)\}$ is a chain sequence. Thus $\xi_1 \geqslant 0$ and the corollary shows the moment problem is determined.

The condition that $\prod m_k(x)/m_k(a)$ converge is difficult to apply in specific cases so we next obtain less general but more readily applicable criteria.

For fixed $a < x \leqslant \xi_1$, set

$$\delta_n = m_n(x) - m_n(a), \qquad \Delta_n = \alpha_n(x) - \alpha_n(a).$$

Then

$$\Delta_n = \delta_n - m_n(x)\,\delta_{n-1} - m_{n-1}(a)\,\delta_n;$$

hence

$$\delta_n = \frac{\Delta_n}{1 - m_{n-1}(a)} + \frac{m_n(x)}{1 - m_{n-1}(a)} \delta_{n-1},$$

$$\frac{\delta_n}{m_n(a)} = \frac{\Delta_n}{\alpha_n(a)} + \frac{m_n(x)\, m_{n-1}(a)}{\alpha_n(a)} \frac{\delta_{n-1}}{m_{n-1}(a)}.$$

Thus

$$\frac{\delta_n}{m_n(a)} = \sum_{k=0}^{n} b_{nk} B_{n-k}, \tag{3.5}$$

where

$$B_k = \frac{\Delta_k}{\alpha_k(a)} \qquad (1 \leqslant k \leqslant n), \qquad B_0 \text{ is immaterial}$$

$$b_{nk} = \frac{m_{n-k}(a) \cdots m_{n-1}(a)\, m_{n-k+1}(x) \cdots m_n(x)}{\alpha_{n-k+1}(a) \cdots \alpha_n(a)} \qquad (1 \leqslant k \leqslant n),$$

$$b_{n0} = 1.$$

Then we also have

$$b_{nk} = \frac{m_{n-k}(a)\, m_{n-k+1}(x) \cdots m_{n-1}(x)\, m_n(x)}{[1 - m_{n-k}(a)] \cdots [1 - m_{n-1}(a)]\, m_n(a)}.$$

Since $m_i(a) \leqslant m_i(x)$ for $a < x \leqslant \xi_1$,

$$0 \leqslant b_{nk} \leqslant \frac{m_{n-k}(x) \cdots m_{n-1}(x)}{[1 - m_{n-k}(x)] \cdots [1 - m_{n-1}(x)]} \cdot \frac{m_n(x)}{m_n(a)} \tag{3.6}$$

for $a < x \leqslant \xi_1$, $0 \leqslant k \leqslant n$.

THEOREM 4. *Let $\xi_1 > -\infty$ and $\sum_n' c_n^{-1}$ converge. If there exists $x < \xi_1$ such that*

$$m_n(x) \leqslant r < \tfrac{1}{2} \qquad (n \geqslant 0); \tag{3.7}$$

then $\sum_i' \xi_i^{-1}$ converges.

Proof. Since $c_n \to +\infty$, then $\alpha_n(x)/\alpha_n(a) \to 1$ $(n \to \infty)$. Hence if $a < x < \xi_1$ and (3.7) holds, then $\{[1 - m_{n-1}(x)]/[1 - m_{n-1}(a)]\}$ is bounded away from 0, hence $\{m_n(x)/m_n(a)\}$ is bounded. Then (3.6) and (3.7) imply

$$0 \leqslant b_{nk} \leqslant MR^k, \qquad 0 \leqslant k \leqslant n,$$

where $M > 0$ and $0 < R = r(1 - r)^{-1} < 1$. Hence by (3.5),

$$\frac{\delta_n}{m_n(a)} \leqslant M \sum_{k=0}^{n} R^k B_{n-k}, \qquad a < x < \xi_1.$$

Moreover,

$$B_j = \frac{\Delta_j}{\alpha_j(a)} = \frac{\alpha_j(x)}{\alpha_j(a)} - 1$$

$$= \frac{(c_j - a)(c_{j+1} - a)}{(c_j - x)(c_{j+1} - x)} - 1.$$

Now the convergence of $\sum' c_n^{-1}$ implies the convergence of $\prod (c_j - a)/(c_j - x)$, hence $\sum B_j$ converges. Therefore we conclude

$$0 < \sum_{n=1}^{\infty} \frac{\delta_n}{m_n(a)} \leqslant \frac{M}{1-R} \sum_{j=0}^{\infty} B_j < \infty.$$

Since $m_n(x)/m_n(a) = 1 + \delta_n/m_n(a)$, it now follows that $\prod m_n(x)/m_n(a)$ converges. From Theorem 3 we conclude that $\sum \xi_i^{-1}$ converges.

Specializing Theorem 4 yields the following simple criterion:

THEOREM 5. *Let*

$$\lim_{n\to\infty} c_n = +\infty, \qquad \limsup_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} < \frac{1}{4}.$$

Then

$$-\infty < \xi_1 < \xi_2 < \cdots, \qquad \lim_{n\to\infty} \xi_n = +\infty,$$

while $\sum_i' \xi_i^{-1} < \infty$ *if and only if* $\sum_n' c_n^{-1} < \infty$.

Proof. The hypotheses assure us of the existence of an x and an r such that

$$c_n > x, \qquad 0 < \alpha_n(x) \leqslant r < \tfrac{1}{4} \qquad (n \geqslant 1).$$

Thus $\{\alpha_n(x)\}$ is a chain sequence and ξ_1 is finite.

Now $r = (1 - g_{n-1}) g_n$, where

$$g_n = g = \tfrac{1}{2}[1 - (1 - 4r)^{1/2}];$$

hence $m_n(x) \leqslant g < \frac{1}{2}$ $(n \geqslant 0)$. Thus if $\sum' c_n^{-1} < \infty$, then according to Theorem 4, $\sum' \xi_i^{-1} < \infty$ while if $\sum' c_n = \infty$, then $\sum \xi_i^{-1} = +\infty$. But in the latter case, the associated Hamburger moment problem is determined (Corollary to Theorem 3). It then follows from [5, Theorem 8] that $\xi_n \to +\infty$ and hence from Theorem 1 that $\xi_i < \xi_{i+1}$.

We next note that it is possible to have $\sum_i' \xi_i^{-1} < \infty$ when $\limsup \alpha_n(0) = \frac{1}{4}$—in fact, when $\lim \alpha_n(0) = \frac{1}{4}$. (An example of this happening when $\limsup \alpha_n(0) > \frac{1}{4}$ will be given in the next section.)

THEOREM 6. *If* $0 < s \leqslant \alpha_n(x) \leqslant \frac{1}{4}$ *for some* $x (n \geqslant 1)$ *and if* $\lim_{n\to\infty} c_n = +\infty$, $\sum_n' n^2/c_n < \infty$, *then* $\xi_1 > -\infty$ *and* $\sum_i' \xi_i^{-1} < \infty$.

Proof. The minimal parameters of the chain sequence $\{\frac{1}{4}\}$ are $g_n = n/(2n+2)$ while for $0 < r < \frac{1}{4}$, the minimal parameters of $\{r\}$ increase monotonically to $\mu(r) = [1 - (1 - 4r)^{1/2}]/2$ [15, (19.12)].

Now the hypotheses assure us that ξ_1 is finite and also that if $a < x < \xi_1$, then there is a t such that

$$0 < t \leqslant \alpha_n(a) < \alpha_n(x) \leqslant \tfrac{1}{4}.$$

Thus

$$0 < \mu(t) \leqslant m_n(a) < m_n(x) \leqslant \frac{n}{2(n+1)} \qquad (n \geqslant 1).$$

Referring to (3.5) and (3.6), we find that

$$b_{nk} \leqslant \frac{1}{2\mu(t)} \frac{(n-k)(n-k+1)}{(n+1)^2};$$

hence that

$$0 \leqslant \frac{\delta_n}{m_n(a)} \leqslant \frac{[2\mu(t)]^{-1}}{(n+1)^2} \sum_{k=1}^{n} k(k+1) B_k .$$

But

$$\begin{aligned} 0 < B_n &= \frac{\alpha_n(x)}{\alpha_n(a)} - 1 \\ &= (x-a) \left\{ \frac{1}{c_n - x} + \frac{1}{c_{n+1} - x} + \frac{x+a}{(c_n - x)(c_{n+1} - x)} \right\}. \end{aligned}$$

Therefore, if $\sum' n^2/c_n < \infty$, then $\sum k(k+1)B_k < \infty$; hence $\sum \delta_n/m_n(a) < \infty$. Thus $\prod m_n(x)/m_n(a)$ converges and Theorem 3 now shows that $\sum_i' \xi_i^{-1}$ converges.

As mentioned in the introduction, our methods yield no information concerning the case $\xi_1 = -\infty$, $\lim_{n\to\infty} x_{nn} = +\infty$ except in the symmetric case—that is, except for a sequence $\{R_n(x)\}$, satisfying a recurrence of the form

$$\begin{gathered} R_n(x) = xR_{n-1}(x) - \gamma_n R_{n-2}(x), \\ R_{-1}(x) = 0, \qquad R_0(x) = 1, \qquad \gamma_{n+1} > 0 \qquad (n \geqslant 1). \end{gathered} \tag{3.8}$$

In such a case, the preceding theory can be applied to the related sequence $\{P_n(x)\}$, where $P_n(x^2) = R_{2n}(x)$, which satisfies (1.1) with

$$c_n = \gamma_{2n-1} + \gamma_{2n}, \qquad \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1} \qquad (n \geqslant 1; \gamma_1 = 0) \tag{3.9}$$

(see [5]). Without going into details, we cite one simple result concerning the zeros of $\{R_n(x)\}$ derived from Theorem 5.

Denote the positive zeros of $R_n(x)$ by

$$z_{n1} < z_{n2} < \cdots < z_{np}, \qquad p = \left[\frac{n}{2}\right],$$

and put $\zeta_i = \lim_{n\to\infty} z_{2n,i}$, $\zeta_i' = \lim_{n\to\infty} z_{2n+1,i}$. Then $0 < \zeta_i < \zeta_i' < \zeta_{i+1}$.

Now let $\lim_{n\to\infty}(\gamma_{2n-1} + \gamma_{2n}) = \infty$ and let $G = \lim_{n\to\infty} \gamma_{2n-1}/\gamma_{2n}$ exist (possibly ∞). Application of Theorem 5 to (1.1) as determined by (3.9) then shows that if $G \neq 1$, then $\lim_{n\to\infty} \zeta_n = \infty$; so $\zeta_i < \zeta_{i+1}$ and the set Z (see (2.1)) for $\{R_n(x)\}$ consists of 0, $\pm \zeta_i$, $\pm \zeta_i'$ $(i \geqslant 1)$. Moreover, $\sum \zeta_n^{-1} < \infty$ if and only if $\sum (\gamma_{2n-1} + \gamma_{2n})^{-1} < \infty$.

It can also be shown using results from [5] that if the Hamburger moment problem associated with $\{P_n(x)\}$ is determined, then for $G < 1$, $\zeta_1 > 0$ and $\zeta_i' = \zeta_i$, while for $G > 1$, $\zeta_1 = 0$, and $\zeta_i' = \zeta_{i+1}$.

IV. Examples and Remarks

Carlitz [4] has studied four sets of orthogonal polynomials which illustrate Theorem 5. The first of these satisfies the recurrence

$$F_{n+1}(x) = [x - a(2n+1)^2]\,F_n(x) - k^2(2n-1)\,(2n)^2\,(2n+1)\,F_{n-1}(x),$$

where $a = k^2 + 1$ and $0 < k^2 < 1$. Carlitz shows the $F_n(x)$ are orthogonal with respect to a distribution function whose spectrum consists of the points

$$s_k = C(2k+1)^2 \qquad (k = 0, 1, 2,...)$$

(C being a certain positive constant whose value he gives).

Since here

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = \frac{k^2}{(k^2+1)^2} = L < \frac{1}{4},$$

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1 < \frac{1}{k^2} = \frac{1-\mu}{\mu},$$

where $\mu = [1 - (1-4L)^{1/2}]/2$, it follows from [6, Theorem 4.3] that the associated Hamburger moment problem is determined. Therefore we can conclude that $\xi_k = s_k$ and $\sum \xi_k^{-1} < \infty$ as predicted.

The remaining three examples are very similar and we will not discuss them. However, a second example worth noting is provided by the "q-polynomials" of Al-Salam and Carlitz [1] for which

$$c_{n+1} = (1+a)\,q^{-n}, \qquad \lambda_{n+1} = aq^{1-2n}(1-q^n), \qquad a > 0, \qquad 0 < q < 1.$$

These polynomials are shown to be orthogonal with respect to a distribution function whose spectrum consists of

$$t_k = q^{-k} \qquad (k = 0, 1, 2,...)$$

(at least when $aq < 1$). The Hamburger moment problem associated with these polynomials is determined if and only if $0 < a \leqslant q < 1$ or $1 < q^{-1} \leqslant a$ [8, p. 483]. Thus, at least for $a \leqslant q$, we must have $\xi_k = t_k$ (it can be shown that $\xi_k = t_k$ when $q < a \leqslant 1$ also—see [8, p. 484]).

Comparison of c_n with ξ_n in these examples suggests that the relationship between $\{c_n\}$ and $\{\xi_n\}$ is much closer than that predicted by Theorem 5. We confess to having no insight into this matter at present.

Still another example where $\sum_n c_n^{-1}$ and $\sum_i \xi_i^{-1}$ both converge is furnished by the Stieltjes–Wigert polynomials (see [5, p. 32]) which are known to be associated with an *indeterminate* moment problem. For these polynomials, however, the ξ_i appear to be unknown.

We know of no examples in the literature in which $\sum_n' c_n^{-1} < \infty$ but $\sum_i' \xi_i^{-1} = +\infty$, but we can construct some from existing examples. The polynomials satisfying

$$M_{n+1}(x) = [x - \beta(2n + \alpha)]\, M_n(x) - (1 + \beta^2)\, n(n + \alpha - 1)\, M_{n-1}(x)$$
$$\beta \text{ real}, \qquad \alpha > 0$$

were first studied by Meixner [9] and later independently by Pollaczek [10]. These polynomials are orthogonal over $(-\infty, \infty)$ with respect to a weight function w which is everywhere positive. Carleman's criterion shows that the associated moment problem is determined.

If we take $\beta = 0$, we obtain the symmetric case (see Carlitz [3] for additional properties in this case when α is an integer). Thus if $P_n(x^2) = M_n(x)$, then $\{P_n(x)\}$ satisfies (1.1) with coefficients given by (3.9) where $\gamma_{n+1} = n(n + \alpha - 1)$, and are orthogonal with respect to the weight function $x^{-1/2}w(x^{1/2})$ over $(0, \infty)$. Thus for $\{P_n(x)\}$, we have $\sum c_n^{-1} < \infty$. However, since the Hamburger moment problem associated with $\{M_n(x)\}$ is determined, then at least the Stieltjes moment problem corresponding to $\{P_n(x)\}$ must be determined. Thus $\xi_i = 0$ $(i \geqslant 1)$.

A second example of this type can be constructed using, essentially, the polynomials $\Omega_n^{(\lambda)}(x)$ considered by Carlitz [3] (and briefly by Stieltjes [12]).

In the above example, we have $\lim \alpha_n(0) = \frac{1}{4}$. An example for which $\limsup \alpha_n(0) > \frac{1}{4}$ and $\sum \xi_i^{-1} < \infty$ can be artificially constructed by considering $\beta_n = (1 - g_{n-1})\, g_n$, where

$$g_0 = 0, \qquad g_{2n} = \frac{1}{n+3}, \qquad g_{2n+1} = r, \qquad \frac{1}{4} < r < \frac{1}{2}.$$

Now choose $c_n > 0$ such that $\sum_n c_n^{-1} < \infty$ and take $\lambda_{n+1} = \beta_n c_n c_{n+1}$. Then $\alpha_n(x) \leqslant \beta_n$ for $x \leqslant 0$; so $\xi_1 \geqslant 0$ and $m_n(x) < g_n \leqslant r$ for $x < 0, n \geqslant 1$. Then by Theorem 4, $\sum' \xi_i^{-1} < \infty$ although $\alpha_{2n+1}(0) \to r > \frac{1}{4}$.

In this last example, $\lim_{n\to\infty} \alpha_n(0)$ does not exist. But in practically all specific examples studied in the literature, the limits

$$c = \lim_{n\to\infty} c_n, \qquad \lambda = \lim_{n\to\infty} \lambda_n, \qquad L = \lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}}$$

exist (possibly ∞). If c and λ are both finite, then Blumenthal [2] has shown that the set X of zeros is dense in $[\sigma, \tau]$, $\sigma = c - 2\lambda^{1/2}$, $\tau = c + 2\lambda^{1/2}$.

If $c = +\infty$, then Theorem 5 gives information about the case $L < \frac{1}{4}$. When $L = \frac{1}{4}$, various possibilities arise as the preceding examples and Theorem 6 show. More examples, not to mention theorems, are needed here. If $c_n = 0(n^2)$, ξ_1 finite (and $L = \frac{1}{4}$), is it the case that $\sigma = \lim \xi_i$ is always finite (so that X is dense in (σ, ∞) [7, Theorem 2])? The examples and our unsuccessful attempts at counter examples lead us to make a half-hearted conjecture to this effect.

Finally, if $L > \frac{1}{4}$ or c is finite and $\lambda = \infty$ then the true interval of orthogonality is $(-\infty, \infty)$ (this can also occur when $L = \frac{1}{4}$ as $\{M_n(x)\}$ shows when $\beta \neq 0$). In this case, we have no results and we hope someone will devise a new method of attack to get information for this situation.

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